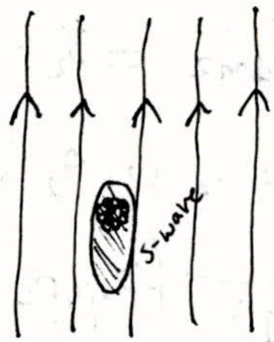
Discussion Notes #9 Ch 4 Griffiths (Also Week 10)

- Focus: Insulators / Dielectrics

Atomic polarizability $\vec{p} = \alpha \vec{E}$

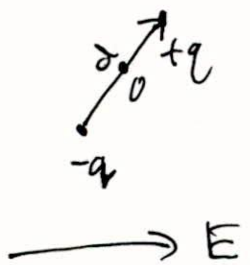
$\vec{p}$ is the dipole moment: $\vec{p} = \int \vec{r}' \rho(r') \, d^3r'$

and $V_{dip} = \dfrac{p\,\hat{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$



For electron cloud model: $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qd}{a^3} \Rightarrow \boxed{p = qd = 4\pi\varepsilon_0 a^3 E}$

Dipole in uniform field:

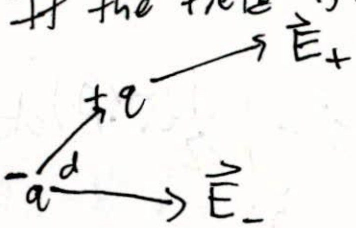


$$\vec{\tau} = \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_-$$

$$\vec{\tau} = \left(\frac{\vec{d}}{2}\right) \times q\vec{E} + \left(\frac{-\vec{d}}{2}\right) \times (-q\vec{E})$$

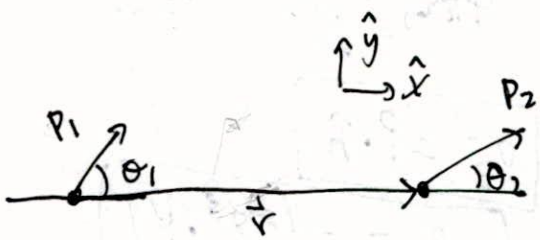
$$\boxed{\vec{\tau} = q\vec{d} \times \vec{E}} \quad \text{or} \quad \boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

If the field is non-uniform $\vec{F} = F_+ + F_- = q(\Delta\vec{E})$, $\Delta\vec{E} \equiv \vec{E}_+ - \vec{E}_-$



$$\Delta E_x = \nabla(E_x) \cdot d$$

$$\Delta\vec{E} = (d \cdot \nabla) E \Rightarrow \boxed{\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E}}$$



Find torque on $\vec{p}_1$ due to $\vec{p}_2$:

$$\vec{E}_{1on2} = E_{dip}(\vec{r})$$

$$\vec{E}_{dip} = -\nabla V_{dip} = -\vec{\nabla}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right) \text{ if } \vec{p} \parallel \hat{z}.$$

$$\vec{E}_{dip} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\vec{E}_{dip} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$

Using the coordinate free representation of $\vec{E}_{dip}$...

$$\vec{\tau} = \vec{p}_2 \times \vec{E}_{1on2} = \vec{p}_2 \times \left(\frac{1}{4\pi\epsilon_0 r^3} \left(3p_1 \cos\theta_1 \hat{x} - \vec{p}_1 \right) \right)$$

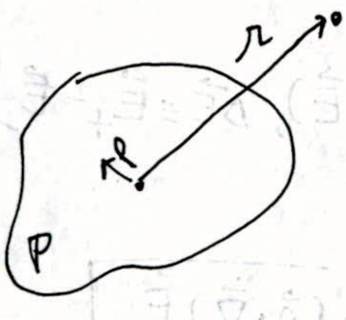
Since $\vec{p}_2 = p_2 \cos\theta_2 \hat{x} + p_2 \sin\theta_2 \hat{y}$

$$\vec{\tau} = \frac{1}{4\pi\epsilon_0 r^3} \left(-3p_1 p_2 \cos\theta_1 \sin\theta_2 - p_1 p_2 \sin(\theta_1 - \theta_2) \right) \hat{z}$$

$$= \frac{-p_1 p_2 \hat{z}}{4\pi\epsilon_0 r^3} \left(3\cos\theta_1 \sin\theta_2 - \sin(\theta_1 - \theta_2) \right)$$

(spot any errors?)

$P \equiv$ dipole moment per unit volume (Polarization)



$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{P(r') \cdot \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \int_V P(r') \cdot \nabla' \left(\frac{1}{\mathscr{r}} \right) d\tau'$$

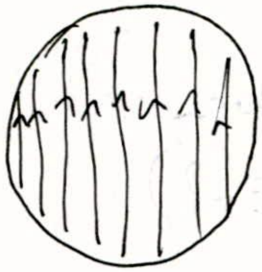
$$= \frac{1}{4\pi\epsilon_0} \left[\int_V \nabla' \cdot \left(\frac{P(r')}{\mathscr{r}} \right) d\tau' - \int_V \frac{1}{\mathscr{r}} (\nabla' \cdot P) \, d\tau' \right]$$

$$= \frac{1}{4\pi\epsilon_0} \oint_S \frac{1}{\mathscr{r}} P \cdot da' - \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{\mathscr{r}} (\nabla' \cdot P) \, d\tau'$$

First term looks like a surface charge $\boxed{\sigma_B = P \cdot \hat{n}}$

Second " volume charge $\boxed{\rho_B = -(\nabla' \cdot P)}$.

e.g. Field produced by uniformly polarized sphere of radius R.



$$\rho_b = 0 \qquad \sigma_b = \vec{P}\cdot\hat{r} = P\cos\theta$$

$\nabla^2 V = 0$ both inside & outside

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\sigma_b = -\varepsilon_0\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\Big|_R = P\cos\theta = P P_1(\cos\theta)$$

$$+\varepsilon_0\left(\sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+2}} P_l(\cos\theta)(l+1)\right)$$

$$= \varepsilon_0\left(\sum_{l=0}^{\infty} P_l(\cos\theta)\left[l A_l R^{l-1} + (l+1)\frac{B_l}{R^{l+2}}\right]\right)$$

Also, from continuity

$$0 = \sum_{l=0}^{\infty}\left(A_l R^l - \frac{B_l}{R^{l+1}}\right) P_l(\cos\theta)$$

$$\Rightarrow A_l R^{2l+1} = B_l \;\forall l \;\;\&\;\; l A_l R^{2l+1} = (l+1) B_l \;\; \forall l \neq 1$$

$$\Rightarrow \boxed{A_l = B_l = 0 \quad \forall l \neq 1}$$

$$\frac{P}{\varepsilon_0} = A_1 R + 2 A_1 R \Longrightarrow \boxed{A_1 = \frac{P}{3\varepsilon_0}} \qquad \boxed{B_1 = \frac{PR^3}{3\varepsilon_0}}$$

$$\boxed{V(r,\theta) = \begin{cases} \frac{P}{3\varepsilon_0} r\cos\theta & r<R \\ \frac{PR^3}{3\varepsilon_0 r^2}\cos\theta & r>R \end{cases}}$$

Same as field of point dipole at origin $r>R$!

Displacement field D

P = field due to bound charges within Dielectric

$\rho = \rho_b + \rho_f$

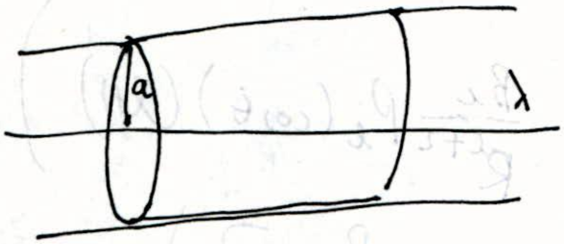
← everything else = ions, electrons on conductor...

$\epsilon_0 \nabla\cdot E = \rho_b + \rho_f = -\nabla\cdot P + \rho_f \qquad \rho_f = \nabla\cdot(\epsilon_0 E + P)$

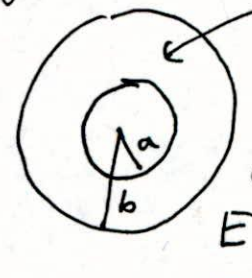
$\boxed{\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}}$ w/ $\nabla\cdot D = \rho_f$

e.g. Wire w/ λ, rubber insulation. $\vec{D} = \frac{\lambda}{2\pi s}\hat{s}$

Outside $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$



e.g. Find E-field everywhere. This is a homework problem.



$P(r) = \frac{k}{r}\hat{r}. \quad \rho_b = -\nabla\cdot P = \frac{-1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) = \frac{-k}{r^2}$

and $\sigma_b = P\cdot\hat{n} = \begin{cases} -k/a & r=a \\ +k/b & r=b. \end{cases}$

Given the bound charges, find E.

$E = 0 \quad r<a$

$E = \frac{Q_{enc}}{4\pi\epsilon_0 r^2}\hat{r} \quad r>a.$

Make sure you understand ex 4.5 in Griffiths!

B.C. Same for E i.e. $E^{\parallel}$ cont, $\Delta E^{\perp} \propto \sigma$

$\rightarrow D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$

$D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$

ϵ Find V(0).

Idea: $D = \begin{cases} \frac{Q}{4\pi r^2}\hat{r} & r>a \\ 0 & r<a. \end{cases}$

$\Rightarrow E = \begin{cases} \frac{Q\hat{r}}{4\pi\epsilon r^2} & [a,b] \\ \frac{Q\hat{r}}{4\pi\epsilon_0 r^2} & [b,\infty) \end{cases}$

Note $\nabla\times D \neq 0$ always. only $\nabla\times E = 0$.

Linear Dielectrics: $D = \epsilon E$ ← permittivity $\quad \epsilon = K\epsilon_0 \quad K = \epsilon_r$ = relative permittivity