Phys 100A Week 8
Discussion Notes

Laplace's equation in spherical coordinates:

$$\nabla^2 V(r,\theta,\phi) = 0$$

Since $d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\left[\partial_r\left(r^2\sin\theta\,\partial_r V\right) + \partial_\theta\left(\sin\theta\,\partial_\theta V\right) + \partial_\phi\left(\frac{1}{\sin\theta}\partial_\phi V\right)\right] = \cancel{0}$$

$$\nabla^2 V = \frac{1}{r^2}\partial_r\left(r^2\partial_r V\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta V\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 V = 0$$

Suppose $\cancel{V = R(r)\Theta(\theta}$ $V(r,\theta,\phi) = R(r)\,X(\theta,\phi)$

Denote $\frac{\partial R}{\partial r} = R'$ $\quad \frac{\partial}{\partial\theta}X = X_\theta \quad \frac{\partial}{\partial\phi}X = X_\phi$

$$X\frac{1}{r^2}\partial_r\left(r^2\partial_r R\right) + \frac{R}{r^2}\left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,X_\theta\right) + \frac{1}{\sin^2\theta}X_{\phi\phi}\right) = 0$$

Divide by XR/r^2...

$$\underbrace{\frac{1}{R}\partial_r\left(r^2\partial_r R\right)}_{\equiv\ \ell(\ell+1)} + \underbrace{\frac{1}{X}\left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,X_\theta\right) + \frac{1}{\sin^2\theta}X_{\phi\phi}\right)}_{-\ell(\ell+1)} = 0$$

why?

$r^2R'' + 2rR' = \ell(\ell+1)R$ Guess $R = r^\alpha$ then

$$r^\alpha\alpha(\alpha-1) + 2\alpha r^\alpha = \ell(\ell+1)r^\alpha \Rightarrow \alpha(\alpha+1) = \ell(\ell+1)$$
$$\Rightarrow \alpha = \ell,\ -\ell-1$$

The angular part satisfies

$$\sin\theta\, \partial_\theta (\sin\theta\, \partial_\theta X) + \partial_\phi^2 X = -\ell(\ell+1)\sin^2\theta\, X$$

Note: ~~$x \equiv l(\cos\theta)$ then~~ $\cancel{\partial_x \neq \frac{\partial x}{\partial \theta}}$ $\cancel{\partial_\theta = \frac{\partial x}{\partial\theta}\partial_x}$

Suppose X also seperates: $X = \Theta(\theta)\,\Phi(\phi)$

Φ must be periodic in ϕ, so let $\Phi'' = -m^2 \Phi$.

i.e. $\Phi \propto e^{\pm i m \phi}$. Clearly m is upper bounded.

The B.C. imply $m \in \mathbb{Z}$. Thus

$m \in \{-\ell \dots \ell\}$, for $\ell \in \mathbb{Z}$.

$$\sin^2\theta\, \Theta'' + \sin\theta\cos\theta\, \Theta' + (\ell(\ell+1)\sin^2\theta - m^2)\Theta = 0$$

Define $x = \cos\theta$ $\quad \partial_\theta = \frac{\partial x}{\partial \theta}\partial_x = -\sin\theta\, \partial_x$

$$\Theta'' = -\sin\theta\, \partial_x(-\sin\theta\, \partial_x \Theta) = \sin^2\theta\, \partial_x^2 \Theta + \sin\theta\cos\theta \frac{1}{\frac{\partial x}{\partial\theta}} \partial_x \Theta$$

$$= (1-x^2)\partial_x^2 \Theta - x\partial_x \Theta$$

$$\Theta' = -\sin\theta\, \partial_x \Theta$$

$$(1-x^2)\partial_x^2 \Theta - x\partial_x \Theta - x\partial_x \Theta + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)\Theta = 0$$

"Associated Legendre Equation". Solutions are indexed by two integers. Simplification: $V = V(r,\theta)$.

$\Rightarrow m = 0 \Rightarrow X(\theta,\phi) = P_\ell(\cos\theta) \Rightarrow V = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right)P_\ell(\cos\theta)$

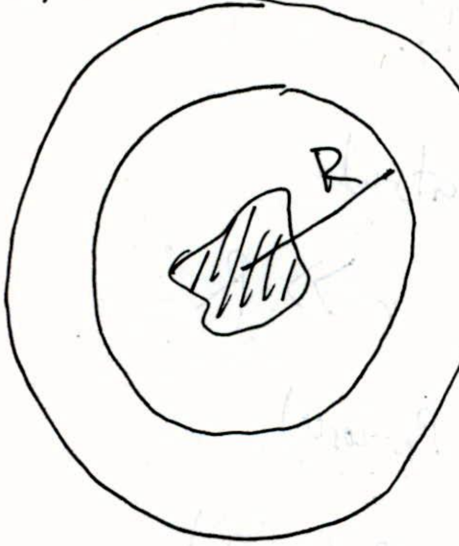
We've surmised for spherically symmetric (azimuthally only) solutions to Laplace's equation

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) Y_{lm}(\theta,\phi) \longrightarrow \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

Can verify $P_0(x) = 1$ $\quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2-1)$

They satisfy $\int_{-1}^{1} P_n(x) P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}$

Aside: For $r > R$ $\nabla^2 V = 0 \Rightarrow$



$$V = \sum_{l=0}^{\infty} \sum_{m=l}^{l} \frac{B_l}{r^{l+1}} Y_{lm}(\theta,\phi)$$

So $V \propto \frac{1}{r}$, $E_r \propto \frac{1}{r^2}$ as promised.

OK time for an example:

$$V(R,\theta) = (1+\sin^2\theta + 2\cos\theta)V_0$$

Find V everywhere:

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \Rightarrow \sum_{l=0}^{\infty} \tilde{A}_l \left(\frac{r}{R}\right)^l P_l(\cos\theta)$$

B.C. says $\sum_{l=0}^{\infty} \tilde{A}_l P_l(\cos\theta) = V_0(1+\sin^2\theta + 2\cos\theta)$

$$V_0\left(1+\sin^2\theta+2\cos\theta\right)=V_0\left(1+(1-x^2)+2x\right)$$

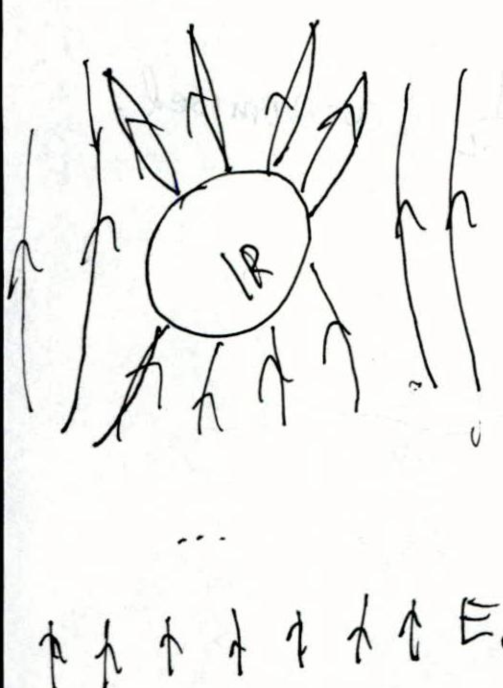
$$=V_0\left(-\frac{2}{3}P_2(x)+2P_1(x)+\frac{5}{3}P_0(x)\right)$$

$$\tilde{A}_2 \qquad \tilde{A}_1 \qquad \tilde{A}_0$$

Thus $V_{in}=V_0\left(\frac{5}{3}P_0(\cos\theta)+2R^{-1}rP_1(\cos\theta)-\frac{2}{3}R^{-2}r^2P_2(\cos\theta)\right)$

Do Vout for practice.

e.g #2. Conductor in static E-field.



$E_0\hat{z}$

$\nabla^2V=0$ inside & outside

$V\to -E_0 r\cos\theta \qquad r\gg R$

$$V_{out}=\sum_{l=0}^{\infty}\left(\frac{B_l}{r^{l+1}}+A_l r^l\right)P_l(\cos\theta)$$

$$V_{in}=\text{const.}=V_0=V_0P_0(\cos\theta)$$

$$V_{out}\Big)_{r=R}=V_0=\sum\left(\tilde{B}_l+\tilde{A}_l\right)P_l(\cos\theta)$$

$$\tilde{B}_0+\tilde{A}_0=V_0$$

$$\tilde{B}_1+\tilde{A}_1=0 \qquad A_1=$$

$$\tilde{B}_i,\tilde{A}_i=0 \qquad i\geq 1$$

<u>Summarizing</u>

$$\tilde{B}_0 + \tilde{A}_0 = V_0 \Leftrightarrow \frac{B_0}{R} + A_0 = V_0$$

$$\frac{B_1}{R^2} + A_1 R = 0 = \frac{B_1}{R^2} - E_0 R$$

Also $\left.\nabla V\right)_R = \left. \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} \right)_R \propto \hat{r}$

$$\left.\frac{\partial V}{\partial \theta}\right)_R = 0 \Rightarrow \frac{B_1}{R^2} + A_1 R = 0 \quad \left(\text{same condition}\right).$$

$$\cancel{V_{out} = A_0 + \frac{(V_0 - A_0)R}{r} - E_0 r\cos\theta + E_0\frac{R^3}{r^2}\cos\theta}$$

$$\cancel{V_{in} = V_0}$$

$$\begin{cases} V_{out} = A_0 + \dfrac{(V_0 - A_0)R}{r} - E_0 r\cos\theta + E_0 \dfrac{R^3}{r^2}\cos\theta \\ V_{in} = V_0 \end{cases}$$

Find $\sigma(\theta)$.