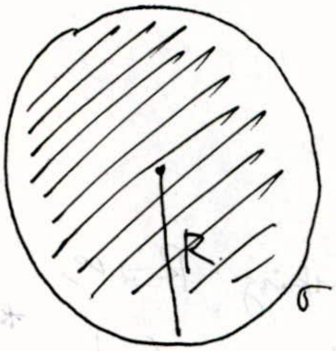
Phys 100A – Discussion 5 Notes

Quiz 2 Review:

Topics: Gauss Law, Electrostatic energy, conductors (Ch 2 Griffiths)

e.g. Calculate the work required to assemble the following charge distribution.

$$\rho(r) = \begin{cases} \rho_0 e^{-r/a} & r < R \\ 0 & r \geq R \end{cases}$$



and $\sigma = \sigma_0 \equiv$ const. On the surface.

Also calculate the force per unit area on the surface.

Recall the potential due to a point charge q

$V = \frac{q}{4\pi\varepsilon_0 r}$. If we have a collection of charges, the energy to assemble them

$$W = \sum_{\substack{\text{pairs} \\ \{i,j\}}} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}} = \frac{1}{2} q_1 \left(\frac{q_2}{4\pi\varepsilon_0 r_{12}} + \frac{q_3}{4\pi\varepsilon_0 r_{13}} \cdots \right) + \frac{1}{2} q_2 \left(\ldots \right) + \ldots$$

$$= \frac{1}{2} \sum_{i=1}^{N} q_i \left(\sum_{j \neq i} \frac{q_j}{4\pi\varepsilon_0 r_{ij}} \right)$$

Generalizing to continuum $\boxed{W = \frac{1}{2} \int dV \, \rho(r) V(r)}$

Using Gauss Law $\rho(r) = \varepsilon_0 \nabla \cdot E$ and

$$\nabla \cdot (f \nabla A) = \nabla f \cdot \nabla A + f \nabla^2 A$$

$$W = \frac{\varepsilon_0}{2} \int d^3r \, (\nabla \cdot E) V = \frac{\varepsilon_0}{2} \int d^3r \, (-\nabla^2 V) V$$

$$= \frac{\varepsilon_0}{2} \int d^3r \left(|\vec{\nabla} V|^2 - \nabla \cdot (V \vec{\nabla} V) \right)$$

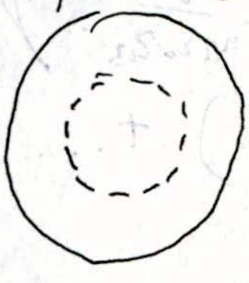
$$\overset{GT}{=} \frac{\varepsilon_0}{2} \left(\int_R d^3r \, E^2 + \int_{\partial R} dS \cdot (\vec{E} V) \right)$$

where R is a region enclosing the charges. Taking $R \to \infty$, if the field dies off at ∞, we can neglect the second term *

$$W = \frac{\varepsilon_0}{2} \int d^3r \, E^2 \qquad \text{where integral is over all space.}$$

Thus to answer question, we should calculate the field everywhere.

→ Spherical gaussian surface



$$E(r)(4\pi r^2) = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_0^r dr' \, r'^2 \int d\Omega \, \rho(r)$$

$$= \frac{4\pi \rho_0}{\varepsilon_0} \int_0^r dx \, x^2 e^{-x/a}$$

$$= \frac{4\pi \rho_0 a^3}{\varepsilon_0} \underbrace{\int_0^{r/a} dx \, x^2 e^{-x}}_{f(r/a)}$$

$$E(r) = \frac{\rho_0 a^3}{\varepsilon_0 r^2} f(r/a)$$

* Fine if your charge distribution is finite. Learn why later.

To calculate the field at $r > R$ we only need Q_{tot}.

$$E(r) = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{Q_{tot}}{4\pi\varepsilon_0 r^2} \quad \text{where} \quad Q_{tot} = 4\pi R^2 \sigma_0 + 4\pi \int_0^R r^2 \rho(r)\, dr$$
$$= 4\pi R^2 \sigma_0 + 4\pi \rho_0 a^3 f(R/a)$$

We are now in a position to calculate $W = \frac{\varepsilon_0}{2} \int E^2 \, dV$

To simplify the calculation $a \to \infty$. Then

$$f(r/a) \longrightarrow \int_0^{r/a} dx \; x^2 = \frac{(r/a)^3}{3}$$

and $$\underset{@\, r<R}{E(r)} \longrightarrow \frac{\rho_0 r}{3\varepsilon_0}$$

Makes sense!

$$\longrightarrow \frac{1}{\cancel{4\pi}\varepsilon_0 r^2}\left(\cancel{4\pi} R^2 \sigma_0 + \cancel{4\pi} \rho_0 \frac{R^3}{3}\right)$$

and

$$W = 4\pi \frac{\varepsilon_0}{2} \int_0^\infty dr\, r^2 E^2 = 2\pi\varepsilon_0 \left(\int_0^R \left(\frac{\rho_0}{3\varepsilon_0}\right)^2 r^4 \, dr + \int_R^\infty \left(\frac{R^2\sigma_0 + \frac{1}{3}R^3\rho_0}{\varepsilon_0}\right)^2 \frac{1}{r^2}\, dr \right)$$

Simplify on board.

Recall formula for $f = \sigma_0 \frac{1}{2}\left(E_{Above} + E_{Below}\right) = \frac{1}{2}\sigma_0 \frac{1}{\varepsilon_0}\left(\sigma_0 + \frac{2}{3}\rho_0 R\right)\hat{r}$.

and $\vec{E}_{above} = \frac{1}{\varepsilon_0 R^2}\left(R^2\sigma_0 + \rho_0 \frac{R^3}{3}\right)\hat{r} = \frac{1}{\varepsilon_0}\left(\sigma + \frac{1}{3}\rho_0 R\right)\hat{r}$

$\vec{E}_{below} = \frac{\rho_0 R}{3\varepsilon_0}\hat{r}$