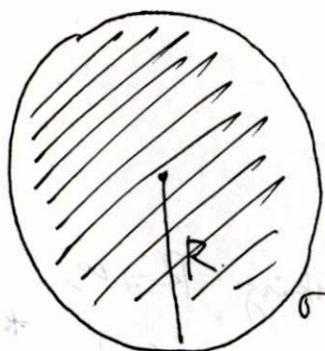


Quiz 2 Review:

Topics: Gauss Law, Electrostatic energy, conductors (Ch 2 Griffiths)

e.g. Calculate the work required to assemble the following charge distribution.

$$\rho(r) = \begin{cases} \rho_0 e^{-r/a} & r < R \\ 0 & r \geq R \end{cases}$$



and  $\sigma = \sigma_0 \equiv \text{const.}$  on the surface.

Also calculate the force per unit area on the surface.

Recall the potential due to a point charge  $q$

$$V = \frac{q}{4\pi\epsilon_0 r}$$

If we have a collection of charges, the energy to assemble them

$$W = \sum_{\text{pairs } \{i,j\}} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} = \frac{1}{2} q_1 \left( \frac{q_2}{4\pi\epsilon_0 r_{12}} + \frac{q_3}{4\pi\epsilon_0 r_{13}} + \dots \right) + \frac{1}{2} q_2 (\dots) + \dots$$

$$= \frac{1}{2} \sum_{i=1}^N q_i \left( \sum_{j \neq i} \frac{q_j}{4\pi\epsilon_0 r_{ij}} \right)$$

Generalizing to continuum

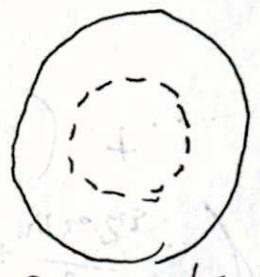
$$W = \frac{1}{2} \int dV \rho(r) V(r)$$

\* Fine if your charge distribution is finite. Learn why later.

$$E(r) = \frac{\rho_0 a^3}{\epsilon_0 r^2} f(r/a)$$

$$= \frac{4\pi \rho_0 a^3}{\epsilon_0} \int_{r/a}^{\infty} dx x^2 e^{-x} f(x/a)$$

$$= \frac{4\pi \rho_0 a^3}{\epsilon_0} \int_0^{\infty} dx x^2 e^{-x/a}$$



$$E(r) (4\pi r^2) = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_r^{\infty} dr' r'^2 \int d\Omega \rho(r')$$

→ Spherical gaussian surface

Thus to answer question we should calculate the field everywhere.

$$W = \frac{\epsilon_0}{2} \int d^3r E^2$$

where integral is over all space.

Where R is a region enclosing the charges. Taking  $R \rightarrow \infty$ , if the field dies off at  $\infty$ , we can neglect the second term.

$$\stackrel{GT}{=} \frac{\epsilon_0}{2} \left( \int d^3r E^2 + \int_{\partial R} dS \cdot (\underline{E} \cdot \underline{V}) \right)$$

$$= \frac{\epsilon_0}{2} \int d^3r (|\underline{\nabla} V|^2 - \nabla \cdot (V \underline{\nabla} V))$$

$$W = \frac{\epsilon_0}{2} \int d^3r (\nabla \cdot \underline{E}) V = \frac{\epsilon_0}{2} \int d^3r (-\nabla^2 V) V$$

$$\nabla \cdot (f \underline{\nabla} A) = \nabla f \cdot \underline{\nabla} A + f \nabla^2 A$$

Using Gauss law  $\rho(r) = \epsilon_0 \nabla \cdot \underline{E}$  and

To calculate the field at  $r > R$  we only need  $Q_{tot}$ .

$$E(r) = \frac{Q_{enc}}{4\pi\epsilon_0 r^2} = \frac{Q_{tot}}{4\pi\epsilon_0 r^2} \quad \text{where } Q_{tot} = 4\pi R^2 \sigma_0 + 4\pi \int_0^R r^2 \rho(r) dr$$

$$= 4\pi R^2 \sigma_0 + 4\pi \rho_0 a^3 f(R/a)$$

We are now in a position to calculate  $W = \frac{\epsilon_0}{2} \int E^2 dV$

To simplify the calculation  $a \rightarrow \infty$ . Then

$$f(r/a) \rightarrow \int_0^{r/a} dx x^2 = \frac{(r/a)^3}{3}$$

and  $E(r) \rightarrow \frac{\rho_0 r}{3\epsilon_0}$   
 @  $r < R$

Makes sense!

$$\rightarrow \frac{1}{4\pi\epsilon_0 r^2} \left( 4\pi R^2 \sigma_0 + 4\pi \rho_0 \frac{R^3}{3} \right)$$

and

$$W = 4\pi \frac{\epsilon_0}{2} \int_0^\infty dr r^2 E^2 = 2\pi\epsilon_0 \left( \int_0^R \left( \frac{\rho_0}{3\epsilon_0} \right)^2 r^4 dr + \int_R^\infty \left( \frac{R^2 \sigma_0 + \frac{1}{3} R^3 \rho_0}{\epsilon_0} \right)^2 \frac{1}{r^2} dr \right)$$

Simplify on board.

Recall formula for  $f = \sigma_0 \frac{1}{2} (E_{above} + E_{below}) = \frac{1}{2} \sigma_0 \frac{1}{\epsilon_0} \left( \sigma_0 + \frac{2}{3} \rho_0 R \right) \hat{r}$

and  $\vec{E}_{above} = \frac{1}{\epsilon_0 R^2} \left( R^2 \sigma_0 + \rho_0 \frac{R^3}{3} \right) \hat{r} = \frac{1}{\epsilon_0} \left( \sigma_0 + \frac{1}{3} \rho_0 R \right) \hat{r}$

$\vec{E}_{below} = \frac{\rho_0 R}{3\epsilon_0} \hat{r}$

To minimize the error we set the derivative to zero

$$E_{total} = \frac{1}{2} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

where  $\hat{y}_i = \sum_{j=1}^n w_{ij} x_{ij}$

$$\frac{\partial E_{total}}{\partial w_{ij}} = (y_i - \hat{y}_i) x_{ij} = 0$$

we now have a system of equations

$$W = \frac{1}{\sum x_{ij}^2} \sum x_{ij} y_i$$

to find the solution  $w \rightarrow \dots$

we have  $E(w) \rightarrow \dots$

in other words!

$$\frac{\partial E}{\partial w_{ij}} = (y_i - \hat{y}_i) x_{ij} = 0$$

$$\begin{pmatrix} \frac{\partial E}{\partial w_{11}} \\ \vdots \\ \frac{\partial E}{\partial w_{1n}} \end{pmatrix} = \begin{pmatrix} (y_1 - \hat{y}_1) x_{11} \\ \vdots \\ (y_1 - \hat{y}_1) x_{1n} \end{pmatrix} + \begin{pmatrix} (y_2 - \hat{y}_2) x_{21} \\ \vdots \\ (y_2 - \hat{y}_2) x_{2n} \end{pmatrix} + \dots + \begin{pmatrix} (y_n - \hat{y}_n) x_{n1} \\ \vdots \\ (y_n - \hat{y}_n) x_{nn} \end{pmatrix}$$

we can write

$$\frac{\partial E}{\partial w_{ij}} = (y_i - \hat{y}_i) x_{ij} = 0$$

$$\frac{\partial E}{\partial w_{ij}} = (y_i - \sum_{k=1}^n w_{ik} x_{ik}) x_{ij} = 0$$

$$\frac{\partial E}{\partial w_{ij}} = (y_i - \sum_{k=1}^n w_{ik} x_{ik}) x_{ij} = 0$$