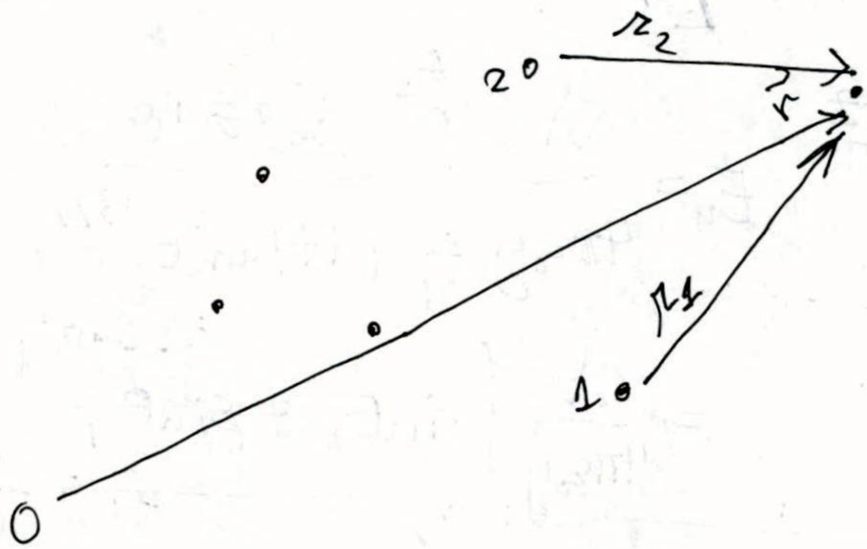
Phys 100A Discussion 4

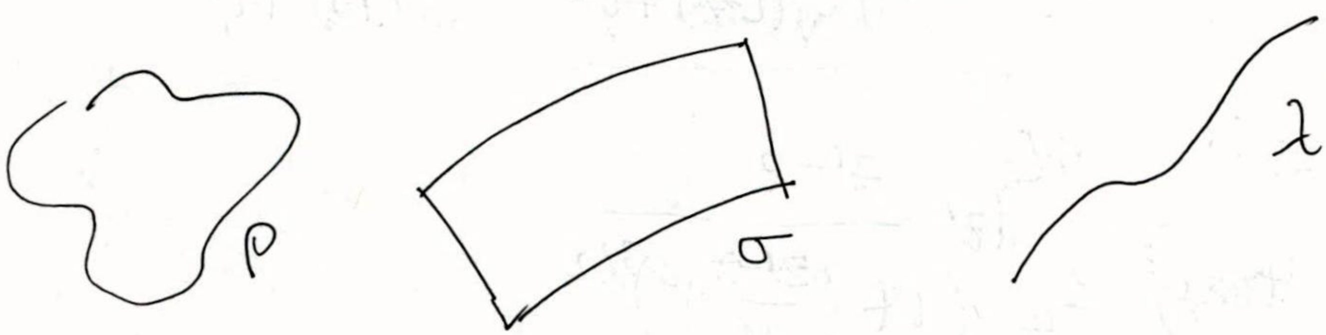
Gauss Law / Charge distributions :

Coulomb Law + Superposition says :

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i$$

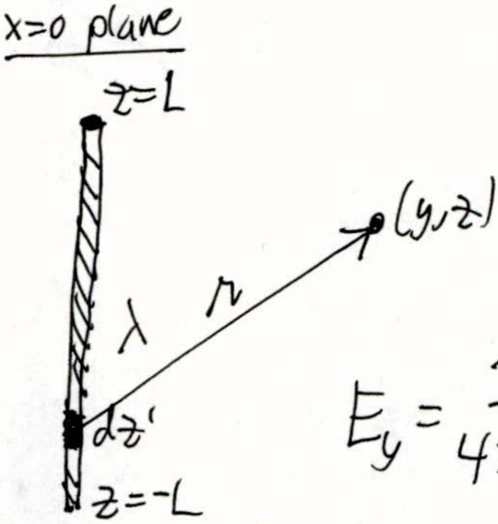


Generalize to cont. charge distributions :



$$\frac{1}{4\pi\varepsilon_0} \int_V d^3r' \, \frac{\rho \, dV'}{r^2} \hat{r} \qquad \frac{1}{4\pi\varepsilon_0} \int_S d^2r' \, \frac{\sigma \, dA'}{r^2} \hat{r} \qquad \frac{1}{4\pi\varepsilon_0} \int_C dl' \, \lambda \, \frac{\hat{r}}{r^2}$$

e.g. Line charge of length $2L$



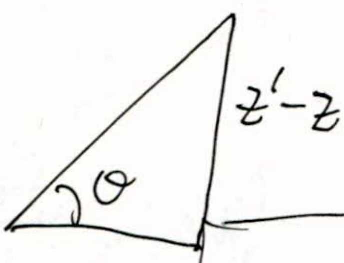
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L} \lambda\, dz' \frac{y\hat{y} + (z-z')\hat{z}}{\left(y^2 + (z-z')^2\right)^{3/2}}$$

$$E_y = \frac{\lambda y}{4\pi\varepsilon_0 y^3}\int_{-L}^{L} dz' \frac{1}{\left(1+\left(\frac{z'-z}{y}\right)^2\right)^{3/2}}$$

Let $\tan\theta = \frac{z'-z}{y}$

$y\sec^2\theta\, d\theta = dz'$

$$E_y = \frac{\lambda}{4\pi\varepsilon_0 y}\int_{\theta_1}^{\theta_2} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$$



$$= \frac{\lambda}{4\pi\varepsilon_0 y}\left[\sin\theta_2 - \sin\theta_1\right]$$

$$\boxed{E_y = \frac{\lambda}{4\pi\varepsilon_0 y}\left(\frac{L-z}{\sqrt{(L-z)^2+y^2}} + \frac{L+z}{\sqrt{(L+z)^2+y^2}}\right)}$$

$$E_z = \frac{-\lambda}{4\pi\varepsilon_0 y^3}\int_{-L}^{L} dz' \frac{z'-z}{\left(1+\left(\frac{z'-z}{y}\right)^2\right)^{3/2}}$$

$$= \frac{-\lambda}{4\pi\varepsilon_0 y}\int_{\theta_1}^{\theta_2} \frac{\sec^2\theta\tan\theta}{\sec^3\theta}\, d\theta = \frac{\lambda}{4\pi\varepsilon_0 y}\left(\cos\theta_2 - \cos\theta_1\right)$$

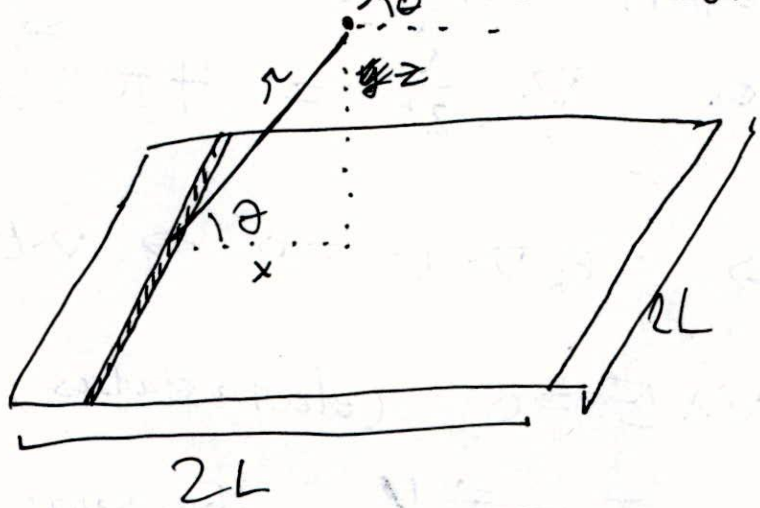
$$\boxed{E_z = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(L-z)^2+y^2}} - \frac{1}{\sqrt{(L+z)^2+y^2}}\right)}$$

e.g. Line charge of length 2L cont.

Generalize outside of plane $\left\{ \begin{array}{l} \hat{y} \longrightarrow \hat{\rho} \\ y \longrightarrow \rho \end{array} \right\}$

Cylindrical symmetry! Generalize to charge above an $2L \times 2L$ sheet, at symmetry point.

$$dE_z = \frac{2\,d\lambda\,L}{4\pi\varepsilon_0 r}\left(\frac{1}{\sqrt{L^2+r^2}}\right)\frac{z}{r}$$



$d\lambda = \sigma\,dx$

$$E_z = \int dE_z = \int_{-L}^{L} dx \, \frac{2\sigma L z}{4\pi\varepsilon_0 \sqrt{x^2+z^2}^{\,2}\sqrt{L^2+x^2+z^2}}$$

$$= \frac{\sigma L z}{2\pi\varepsilon_0}\int_{-L}^{L} dx \, \frac{1}{(x^2+z^2)\sqrt{L^2+x^2+z^2}} \qquad \tan\theta = \frac{\sqrt{x^2+z^2}}{L}$$

In the limit $L \to \infty$

$$E_z \longrightarrow \frac{\sigma z}{2\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dx}{x^2+z^2} = \frac{\sigma}{2\varepsilon_0}$$

Some remarks on Gauss Law

$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$$

E-field defines charge, charge + current define E-field.

Have to be careful where we have singularities i.e. $\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi \delta^{(3)}(\vec{r} - \vec{r}')$

Where $\rho = 0 \rightarrow \varepsilon_0 \nabla \cdot E = 0 \Rightarrow \nabla \cdot E = 0$

By assumption, $\nabla \times E = 0$ (electrostatics)

$\nabla \times E = 0 \iff E = -\vec{\nabla} V$ for some V.

Then $\nabla^2 V = 0$. V is uniquely defined, given $V\big|_{\partial D}$. Known as uniqueness thm.