Phys 100A Discussion Session #3

#1 Practice w/ Levi-Civita notation & Integrals

1.36 Griffiths

(a). Show $\int_S f(\nabla\times A)\cdot dA = \int_S [A\times \nabla f]\cdot dA$
$+ \int_P f A\cdot dl$

(b). Show

$$\int_V B\cdot(\nabla\times A)d\tau = \int_V A\cdot(\nabla\times B)d\tau + \oint_S (A\times B)\cdot dA$$

(a) Want to use Stokes' theorem...

$$\int_P (f\vec{A})\cdot dl = \int_S \nabla\times(f\vec{A})\cdot dA$$

what is $\nabla\times(f\vec{A})$?

$$[\nabla\times fA]_i = \varepsilon_{ijk}\,\partial_j(fA_k) = \varepsilon_{ijk}(\partial_j f)A_k + f\varepsilon_{ijk}\partial_j A_k$$

$$\Rightarrow \nabla\times fA = \nabla f\times A + f\nabla\times A$$

$$\int_P (f\vec{A})\cdot dl = \int_S (\nabla f\times A)\cdot dA + \int_S f(\nabla\times A)\cdot dA \checkmark$$

(b). Apply divergence theorem ...

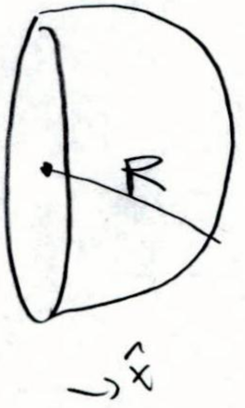
$$\oint_S (A\times B)\cdot da = \int_V \nabla\cdot(A\times B)\,d\tau$$

$$\nabla\cdot(A\times B) = \varepsilon_{ijk}\partial_i(A_j B_k) = \varepsilon_{ijk}(\partial_i A_j)\,B_k + \varepsilon_{ijk} A_j(\partial_i B_k)$$

$$\Rightarrow \quad \nabla\cdot(A\times B) = (\nabla\times A)\cdot B - (\nabla\times B)\cdot A$$

$$\oint_S (A\times B)\cdot da = \int_V (\nabla\times A)\cdot B\,d\tau - \int_V (\nabla\times B)\cdot A\,d\tau \quad ✓$$

Practice w/ Divergence thm :



$$\vec{F}(\vec{x}) = \tfrac{1}{2}x^2\,\hat{x}.$$

First, let's do volume part:

$$\int_V (\nabla\cdot F)\,d\tau = \int_V x\,d\tau$$

Since spherical integral switch to spherical coord.

$$x = r\sin\theta\,\cos\phi$$

$$\int_V x\,d\tau = \int_0^R r^3\,dr \underbrace{\int_0^\pi \sin^2\theta\,d\theta}_{\pi/2} \underbrace{\int_{-\pi/2}^{\pi/2} d\phi\,\cos\phi}_{2}$$

$$= \pi R^4/4$$

Practice w/ divergence thm cont.

Surface part.

$$\int_S F\cdot dA = \cancel{\int_{\text{plane}} F\cdot dA}^{\to 0} + \int_{\text{hemisphere}} F\cdot dA$$

$$= \int_{\text{hemisphere}} F\cdot\hat{r}\,(R^2\sin\theta)\,d\theta\,d\phi$$

$$= R^2\int_{hs} \tfrac{1}{2}(R\sin\theta\cos\phi)^2\,\hat{x}\cdot\hat{r}\,\sin\theta\,d\theta\,d\phi$$

$$\hat{x}\cdot\hat{r} = \cos\phi\sin\theta$$

$$= \tfrac{1}{2}R^4 \underbrace{\int_0^{\pi} d\theta\,(\sin^4\theta)}_{3\pi/8}\ \underbrace{\int_{-\pi/2}^{\pi/2} \cos^3\phi\, d\phi}_{\frac{4}{3}}$$

check

Practice w/ δ - function:

$$\int_{-\infty}^{\infty} e^{ix}\,\delta(3x^2+39x+126)\,dx$$

$$=\int_{-\infty}^{\infty} e^{ix}\,\delta(3(x^2+13x+42))\,dx$$

$$=\int_{-\infty}^{\infty} e^{ix}\,\delta(3(x+6)(x+7))\,dx$$

Recall $\int_A f(x)\,\delta(g(x))\,dx$

$$=\sum_{\substack{x_0\in A,\\ g(x_0)=0}} f(x_0)/|g'(x_0)|$$

$$x_0=\{-6,-7\}.$$

$$g'(-6)=6(-6)+39=3$$

$$\int_{-\infty}^{\infty} e^{ix}\,\delta(\ldots)\,dx \quad \frac{e^{-6i}+e^{-7i}}{3}=\frac{2}{3}e^{-6.5i}\cos(0.5)$$