Phys 100A Week 2 Discussion

Warm-up:

Simplify $(d=3)$:

$\varepsilon_{ijk}\,\varepsilon_{ij'k'} = ?$

Reminder

$$\varepsilon_{ijk} = \begin{cases} \mathrm{sgn}(ijk) & (i,j,k) = \pi(1,2,3). \\ 0 & \text{otherwise} \end{cases}$$

Given i, only terms that contribute are fixed.

e.g. $i=1$, $j,k \in \{2,3\}$ $j \neq k$.

Given j,k, only one value of i contributes to non-zero sum. If $j=k$ or $j'=k'$, we get zero.

e.g. $j=2$, $k=3 \Rightarrow i=1$.

Only terms that contribute are where

$j=j'$, $k=k'$ OR $j=k'$ $k=j'$

$$\varepsilon_{ijk}\,\varepsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{kj'}\delta_{jk'}$$

Practice w/ ∇, $\nabla\cdot$, $\nabla\times$

$\vec{\nabla}\frac{1}{r} = ?$ (do in cartesian and spherical).

cartesian:

$$\vec{\nabla}\frac{1}{\sqrt{x^2+y^2+z^2}} = \begin{pmatrix} \frac{-x}{(x^2+y^2+z^2)^{3/2}} \\ \vdots \end{pmatrix} = -\vec{r}/r^3 = -\frac{\hat{r}}{r^2}$$

Spherical:

$$\vec{\nabla}\frac{1}{r} = \frac{\partial r^{-1}}{\partial r}\hat{r} = -\frac{1}{r^2}\hat{r} \quad ✓$$

$$\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = ?$$

Use product rule!

$$(\vec{\nabla}\cdot\vec{r})^1/_{r^3} + (\nabla^1/_{r^3})\cdot\vec{r} = ?$$

Cartesian: $\vec{\nabla}\cdot\vec{r} = 3 \qquad \nabla\frac{1}{r^3} = -3\vec{r}/r^5$

$$\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 0 \qquad r > 0.$$

Spherical: $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r^2\sin\theta}\left[\partial_r\left(r^2 \cancel{\sin\theta}\frac{1}{r^2}\right)\right] = 0$ ✓

$$\cancel{\vec{\nabla}\cdot\vec{r} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, r)\right]}$$

In HW you show:

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

Let's show:

$$[\nabla \times (\nabla \times A)]_i = \varepsilon_{ijk} \partial_j (\nabla \times A)_k$$

$$= \varepsilon_{ijk} \partial_j (\varepsilon_{kmn} \partial_m A_n)$$

$$= \varepsilon_{ijk} \varepsilon_{kmn} \partial_j \partial_m A_n$$

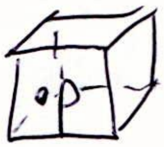
Do sum over k

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \partial_j \partial_m A_n$$

$$= \partial_i (\nabla \cdot A) - \nabla^2 A_i$$

So $\nabla \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$

Some intuition for #2 :

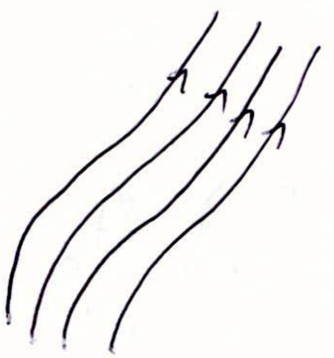


Gauss Law:

$$\int d^3x \, \nabla \cdot F = \int dS \cdot F$$

V tiny $\Rightarrow$ $(\nabla \cdot F) V = \text{Flux} \Rightarrow \nabla \cdot F \approx \dfrac{\text{Flux}}{V}$

OK so what if $\nabla \cdot F = 0$?



e.g. E when $\rho = 0$.

Electric field lines flow

$\Rightarrow$ conserved

$\Rightarrow$ # of them determined by

Gauss law $\nabla \cdot E = \rho / \varepsilon_0$.

Practice w/ Calculus:

$$h(x,y) = -16x^4 + x^2y^2 - 12y^4 + 100$$

top: $\vec{\nabla} h = 0 = \begin{pmatrix} -64x^3 + 2xy^2 \\ -48y^3 + 2yx^2 \end{pmatrix}$

$$\Rightarrow \begin{cases} -64x^2 + 2y^2 = 0 \\ -48y^2 + 2x^2 = 0 \end{cases}$$

$x \neq 0$
$\Rightarrow y = \pm\sqrt{32}\,x$
$x = \pm\sqrt{24}\,y$

$$\Rightarrow \begin{cases} (y - \sqrt{32}\,x)(y + \sqrt{32}\,x) = 0 \\ (x - \sqrt{24}\,y)(x + \sqrt{24}\,y) = 0 \end{cases}$$

(x,y): $\{(0,0)\}$

∇h : points along direction of increase
$|\nabla h|$ is slope along that direction.

Fundamental theorem:

$$\int_C d\vec{l} \cdot \vec{\nabla} f = f(B) - f(A)$$

Check: Let $\cancel{f(x,y,z) = \sinh(x)\sinh(y)\sinh(xyz)}$

$A = 0$

$B = (1,1,1)^T$

$$f(x,y,z) = \sinh(xyz) + z^2$$

$$\cancel{\vec{\nabla} f = \left(\cosh(x)\sinh(y)\sinh(xyz) + \sinh x \sinh y \; yz\cosh x \ldots\right)}$$

$$\nabla f(x,y,z) = \begin{pmatrix} yz\cosh xyz \\ xz\cosh xyz \\ xy\cosh xyz + 2z \end{pmatrix}$$

$C = (0,0,0) \to (1,0,0)$
$(1,0,0) \to (1,1,0)$
$(1,1,0) \to (1,1,1)$

$$\int_A^B \vec{\nabla} f \cdot dl = \cancel{\int_0^1 (\nabla f)_x \, dx + }$$

$$\int_{C_1} \nabla f \cdot dl = 0$$

$$\int_{C_2} \vec{\nabla} f \cdot d\vec{l} = 0$$

$$\int_{C_3} \vec{\nabla} f \cdot d\vec{l} = \int_0^1 dz \left(\cosh z + 2z\right) = \left(\sinh z + z^2\right)\Big|_0^1$$

$$= 1 + \sinh 1$$